

We discuss the gravitational waves (GW) in the context of vector inflation. We derive the action for tensor perturbations and find that tachyonic instabilities are present in most (but not all) of the inflationary models with large fields. In contrast, the stability of the small field inflation ($A_\mu A^\mu \ll \frac{1}{N}$) is ensured by the usual slow-roll conditions, where N is the total number of fields. For example, the Coleman-Weinberg potential and the power-law inflation are always stable in the small fields limit with an approximately flat spectrum of GW. We also provide some examples which lead to a rapid decay of GW and predict the absence of tensor modes in the CMB.

I. INTRODUCTION

In the earlier paper [1] we proposed a new model of inflation in which the quasi de Sitter expansion is driven by vector fields. Isotropy was achieved by employing a triad of mutually orthogonal vector fields or by a large number of randomly oriented fields. The problem of slow-roll was solved by non-minimal coupling of the vector fields to gravity. (Other cosmological models with vector fields have also been recently proposed [2, 3, 4, 5, 6, 7]).

The theory of Ref. [1] is defined by the following action

$$S = \int \sqrt{-g} \left[-\frac{R}{2} \left(1 + \sum_{a=1}^N \frac{1}{6} I_{(a)} \right) - \frac{1}{4} \sum_{a=1}^N F_{\mu\nu}^{(a)} F^{\mu\nu}_{(a)} - \sum_{a=1}^N V(I_{(a)}) \right] dx^4. \quad (1)$$

where

$$I_{(a)} \equiv -A_\mu^{(a)} A_{(a)}^\mu \\ F_{\mu\nu}^{(a)} \equiv \nabla_\mu A_\nu^{(a)} - \nabla_\nu A_\mu^{(a)}$$

and summation over repeated space-time indices is assumed. In the spatially flat Friedmann universe with conformal metric

$$ds^2 = a^2(\eta) (d\eta^2 - \delta_{ik} dx_i dx_k)$$

the evolution of homogeneous background fields is given by

$$A_0 = 0 \\ B''_i + 2\mathcal{H}B'_i + 2\frac{dV(I)}{dI}a^2B_i = 0 \quad (2)$$

where by prime we denote the differentiation with respect to conformal time η and $I = B_i B_i \equiv B^2$, $B_i \equiv \frac{A_i}{a} = -aA^i$, $\mathcal{H} \equiv \frac{a'}{a}$. Equation (2) is equivalent to

$$A''_i + \left(2a^2 \frac{dV}{dI} - \frac{a''}{a} \right) A_i = 0.$$

The Einstein equations reduce to

$$3\mathcal{H}^2 = 8\pi N \left(V(B) a^2 + \frac{1}{2} B'^2 \right), \quad (3)$$

$$2\mathcal{H}' + \mathcal{H}^2 = 8\pi N \left(V(B^2) a^2 - \frac{1}{2} B'^2 \right) \quad (4)$$

and for the usual mass term we have $V = \frac{m^2 B^2}{2} = -\frac{m^2 A_\mu A^\mu}{2}$ and $\frac{dV}{dI} = \frac{m^2}{2}$.

The paper is organized as follows. In the next section we derive the action for gravitational waves. In the third section we quantize the tensor perturbations and solve the corresponding equations of motion. The consequences of our results for different models of vector inflation are analyzed in the forth section and the main conclusions are summarized in the final section.

We consider the transverse and traceless metric perturbations h_{ik} on a spatially flat Friedmann background

$$ds^2 = a(\eta)^2 (d\eta^2 - (\delta_{ik} - h_{ik}) dx_i dx_k)$$

where $h^i_i = 0$ and $h^i_{j,i} = 0$. As the matter content is of vectorial nature, these are the only tensor perturbations of the theory. The corresponding equation of motion can be obtained as a tensor part of the spatial components of Einstein equations. One should note that scalar, vector and tensor perturbations can couple to the background vectors [8]. Some of these couplings vanish due to the background rotational symmetry. For example, $\sum_{a=1}^N h_{ik} A_i^{(a)} A_k^{(a)} = 0$ because it is proportional to the trace of h , which is a consequence of the isotropy condition $\sum_{a=1}^N A_i^{(a)} A_k^{(a)} \propto \delta_{ik}$ and follows from the fact that the trace is the only linear rotational invariant of a matrix. Nevertheless, terms of the form $A_i \delta A_k$ are present in T_i^k and in general do not decouple from GW. However, if we consider random vector fields with random fluctuations, these terms would be statistically suppressed.¹ This allows us to consider the evolution of GW separately from the vector and scalar perturbations.

From now on, in this article, we consider only the tensor perturbations. To obtain the action for these perturbations we must expand Eq. (1) to the second order in h_{ik} . The spatial part of the metric is given by $g_{ij} = -\delta_{ij} + h_{ij}$ and $g^{ij} = -\delta_{ij} - h_{ij} - h_{ik} h_{kj}$ up to the second order. It implies

$$\sqrt{-g} = a^4 \left(1 - \frac{1}{4} h_{ij}^2 \right),$$

$$R = \frac{1}{a^2} \left(-6 \frac{a''}{a} + h_{ij} h_{ij}'' + \frac{3}{4} h_{ij}^2 - h_{ij} \triangle h_{ij} - \frac{3}{4} h_{ij,k}^2 + \frac{1}{2} (h_{ij} h_{ik,j})_{,k} + 3 h_{ij} h_{ij}' H \right)$$

and

$$S_{gw} \approx \frac{1}{8} \int a^2 \left[\left(\frac{1}{8\pi} + \frac{NB^2}{6} \right) (h_{ik}'^2 - h_{ik,j}^2 - m_g^2 h_{ik}^2) \right] dx^3 d\eta \quad (5)$$

(where we used the background equations, averaging and integrations by parts). The graviton mass squared is given by

$$m_g^2 \equiv -16\pi N \frac{\left(\frac{a''}{a^3} - 2V_{,I} - \frac{4}{5} B^2 V_{,II} \right) a^2 B^2 + (B' + B\mathcal{H})^2}{3 + 4\pi N B^2} \quad (6)$$

where $V_{,I} \equiv \frac{dV}{dI}$, $V_{,II} \equiv \frac{d^2 V}{dI^2}$. Unusual mass of the graviton comes from the terms $A_\mu A_\nu g^{\mu\nu}$ and $F_{\mu\nu} F_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta}$. It is proportional to $B^2 H^2$ which remains approximately constant during slow-roll inflation.

The corresponding equation of motion (which could also be derived from the linearized Einstein equation) is

$$h''_{ik} + 2 \left(\mathcal{H} + \frac{4\pi N B B'}{3 + 4\pi N B^2} \right) h'_{ik} - \triangle h_{ik} = -m_g^2 h_{ik}. \quad (7)$$

During the slow-roll inflation $\ddot{B} \ll H\dot{B}$ implies $B'' \approx \mathcal{H}B'$ and we obtain

$$m_g^2 \approx 16\pi N \frac{(-8\pi N V + \frac{10}{3} V_{,I} + \frac{4}{5} B^2 V_{,II}) a^2 B^2}{3 + 4\pi N B^2} \quad (8)$$

which can be further reduced to

$$m_g^2 \approx 16\pi m^2 a^2 N B^2 \left(\frac{5 - 12\pi N B^2}{9 + 12\pi N B^2} \right)$$

for the chaotic potential $V(B^2) = \frac{1}{2} m^2 B^2$. In the original paper [1] we have shown that $B \gtrsim \frac{1}{\sqrt{N}}$ for this potential, which means that the evolution can be unstable due to the tachyonic mass of the graviton $m_g^2 < 0$. We will come back to the stability issue in the forth section after the theory is properly quantized.

¹ Terms of the form $A_i \delta A_k$ are exactly canceled for perturbations which correspond to rotations of a vector triad configuration as a whole.

We follow the procedure outlined in Ref. [9] to quantize the action for GW (5). The first step is to expand the tensor perturbations into Fourier modes

$$h_{ij}(\mathbf{x}, \eta) = \int h_{\mathbf{k}}(\eta) e_{ij}(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}} \frac{d^3k}{(2\pi)^{3/2}}, \quad (9)$$

where $e_{ij}(\mathbf{k})$ is the polarization tensor. The result is substituted in Eq. (5) to obtain

$$S_{gw} \approx \int \frac{1}{8} a^2 e_{ij}^2 \left[\left(\frac{1}{8\pi} + \frac{NB^2}{6} \right) (h'_{\mathbf{k}} h'_{-\mathbf{k}} - (k^2 + m_g^2) h_{\mathbf{k}} h_{-\mathbf{k}}) \right] dk^3 d\eta.$$

It is convenient to introduce a new variable

$$v_{\mathbf{k}} = \frac{1}{2} a h_{\mathbf{k}} \sqrt{e_{ij}^2 \left(\frac{1}{8\pi} + \frac{NB^2}{6} \right)} \quad (10)$$

and rewrite the action as

$$S_{gw} \approx \frac{1}{2} \int [v'_{\mathbf{k}} v'_{-\mathbf{k}} - \omega_k^2(\eta) v_{\mathbf{k}} v_{-\mathbf{k}}] dk^3 d\eta$$

Assuming the slow-roll regime we convert the equation of motion (2) into $B' = -\frac{2a^2 V_{,I} B}{3\mathcal{H}}$ while the Friedman equations (3) and (4) yield $\frac{a''}{a} = \frac{16\pi}{3} N a^2 V$ and one can easily deduce that

$$\omega_k^2(\eta) \equiv k^2 - \frac{a''(\eta)}{a(\eta)} \beta$$

with

$$\beta \equiv 1 + \frac{24\pi N B^2 - \left(\frac{23}{2} \frac{V_{,I}}{V} B^2 + \frac{12}{5} \frac{V_{,II}}{V} B^4 \right)}{4\pi N B^2 + 3}. \quad (11)$$

The corresponding equation of motion is

$$v''_{\mathbf{k}} + \omega_k^2(\eta) v_{\mathbf{k}} = 0. \quad (12)$$

After the usual quantization procedure we obtain the standard power spectrum of the created waves

$$\delta_h^2(k, \eta) = \frac{8 |v_{\mathbf{k}}|^2 k^3}{\pi a^2}. \quad (13)$$

To solve the Eq. (12) during the slow-roll inflation we can use an approximation, where $\beta \approx \text{const}$ and $\frac{a''}{a} \simeq \frac{2}{\eta^2}$. The general solution is given by

$$v_{\mathbf{k}}(\eta) = \sqrt{\eta} \left(C_1 J_{\frac{\sqrt{1+8\beta}}{2}}(k\eta) + C_2 Y_{\frac{\sqrt{1+8\beta}}{2}}(k\eta) \right)$$

where J and Y are Bessel functions of the first and second kind respectively. For the short-wavelength modes with $k^2 \gg \left| \frac{a''}{a} \beta \right|$ we obtain the usual result

$$v_{\mathbf{k}}(\eta) \simeq \frac{1}{\sqrt{k}} e^{\pm i k(\eta - \eta_i)}$$

and for the long-wavelength perturbations with $k^2 \ll \left| \frac{a''}{a} \beta \right|$ the solution becomes

$$v_{\mathbf{k}}(\eta) \simeq C_1 \eta^{\frac{1-\sqrt{1+8\beta}}{2}} + C_2 \eta^{\frac{1+\sqrt{1+8\beta}}{2}}.$$

In the limit of $\beta \rightarrow 1$, the evolution of tensor perturbation is identical to the scalar field inflation. The non-decaying super-horizon modes of GW ($h_{\mathbf{k}} \propto \frac{v_{\mathbf{k}}}{a}$) are frozen, and from Eq. (13) we get an approximately flat power spectrum with a slightly red tilt. However in general the tensor perturbations could grow with time if $\beta > 1$, and decay if $\beta < 1$ which would lead to somewhat different predictions.

Now we are in a position to analyze the behavior of GW in different inflationary scenarios. The key ingredient of our discussion is the expression (11) for β , which can be simplified in the two limiting cases corresponding to the large and small field approximations.

If the inflationary evolution takes place at large values of the field ($B \gg \frac{1}{\sqrt{N}}$), then

$$\beta \approx 7 - \frac{1}{4\pi N} \left(\frac{23}{2} \frac{V_{,I}}{V} + \frac{12}{5} B^2 \frac{V_{,II}}{V} \right). \quad (14)$$

Such models would generically predict $\beta \sim 7$ leading to very large instabilities of GW, incompatible with observations. Nevertheless, for some potentials the second term on the right-hand side of Eq. (14) can be large enough to reduce β towards the observationally allowed range ($\beta \lesssim 1$). Unfortunately, as we shall see, it is rather hard to obtain a working model of vector inflation with large fields.

A much more promising class of models describes the evolution at small values of the inflation field. From Eq. (11), in the limit $B \ll \frac{1}{\sqrt{N}}$, we obtain

$$\beta \approx 1 - \left(\frac{23}{6} \frac{V_{,I}}{V} B^2 + \frac{4}{5} \frac{V_{,II}}{V} B^4 \right). \quad (15)$$

Clearly, all of the models with $\frac{23}{6} \frac{V_{,I}}{V} B^2 + \frac{4}{5} \frac{V_{,II}}{V} B^4 \ll 1$ would predict a stable evolution with nearly flat spectrum of tensor perturbations, similarly to the standard scalar field inflation. In fact, the usual slow-roll conditions

$$\begin{aligned} \frac{V_{,B}}{V} &= 2 \frac{V_{,I}}{V} B \ll 1 \\ \frac{V_{,BB}}{V} &= 2 \frac{V_{,I}}{V} + 4 \frac{V_{,II}}{V} B^2 \ll 1 \end{aligned}$$

automatically imply

$$\begin{aligned} \frac{23}{6} \frac{V_{,I}}{V} B^2 &\ll \frac{23}{6} B < 1 \\ \frac{4}{5} \frac{V_{,II}}{V} B^4 &\ll 1 \end{aligned}$$

in the limit of small fields $B \ll \frac{1}{\sqrt{N}}$ and for a large number of fields $N \gtrsim 15$. However, for a relatively small number of fields $N \lesssim 15$ one should also keep track of the parameter $\frac{23}{6} \frac{V_{,I}}{V} B^2$, which could be of order one.

A. Chaotic potential

Consider a model of chaotic vector inflation ($V = \frac{m^2 B^2}{2}$) proposed in Ref. [1]. It follows from Eq. (14) that during inflation $\frac{1}{\sqrt{2\pi N}} < B < \frac{1}{N^{1/4}}$ and the parameter β stays in between $\beta_i \approx 7$ and $\beta_f \approx 5$. At the beginning of inflation the contribution of the terms $\frac{V_{,I}}{V} B^2$ and $\frac{V_{,II}}{V} B^4$ is suppressed by a factor of $\frac{1}{\sqrt{N}}$, and β changes very slowly down from the value of 7. (An additional problem of the chaotic vector inflation in the spatially curved universes was discussed in Ref. [10].)

The instability can be fixed by going to higher powers of B^2 . For a potential $V = V_0 B^{2n}$ with a very large $n > 300$ we can (in principle) have a large number of e-folds $\mathcal{N} = \frac{2\pi N(B_i^2 - B_f^2)}{n} > 60$ with relatively small (or negative) values of $\beta \lesssim 7 - \frac{6}{5} \frac{n}{N}$, predicting the absence of tensor modes in the CMB. However, such models seem to be rather fine-tuned and we shall not discuss them any further. Instead we will turn our attention to the models with small field inflation which naturally predict a stable evolution of GW.

B. Power-law inflation

An interesting example describes the power-law inflation with potential $V = V_0 \exp(\alpha \sqrt{A_\mu A^\mu}) = V_0 \exp(\alpha \sqrt{B^2})$, where we need $\alpha \leq 2\sqrt{6\pi N}$ for $p \leq -\frac{\epsilon}{3}$. (One could worry that this potential is not regular at $A_\mu = 0$, but in either case one has to modify the potential at small A_μ for the inflation to end.) For this model, only if the inflation takes place at small values of $B \ll \frac{1}{\sqrt{N}}$ then $\beta \sim 1$ and the evolution is stable.

C. Symmetry-breaking potential

For the symmetry-breaking potential $V = \lambda (B^2 - B_0^2)^2$ the evolution starts at some small value of B_i with β_i close to 1 and ends when $p \sim -\frac{\epsilon}{3}$ where $p = \frac{N}{2} (\dot{B}^2 - 2V)$ and $\epsilon = \frac{N}{2} (2V + \dot{B}^2)$. The number of e-folds in the slow-roll and small-fields approximation is given by

$$\mathcal{N} \approx -\pi N B_0^2 \ln \left(\frac{8\pi}{3} N B_0^2 \lambda \right).$$

where the initial value of the field was set by quantum fluctuations to $B_i \sim H \approx B_0^2 \sqrt{\frac{8\pi}{3} N \lambda}$. Apparently it is hard to obtain a long period of inflation with small fields because of the logarithmic dependence on λ . For $N B_0^2 \sim 0.1$ we must have $\lambda \lesssim 10^{-60}$ in order to get at least 60 e-folds of inflation.

D. Coleman-Weinberg potential

A better illustration of the vector inflation is given by the Coleman-Weinberg potential $V = \lambda \left(B^4 \ln \frac{B^2}{B_0^2} - \frac{1}{2} B^4 + \frac{1}{2} B_0^4 \right)$, where a large number of e-folds can naturally occur at small values of the field ($B < B_0 \ll \frac{1}{\sqrt{N}}$). Indeed, the number of e-folds is

$$\mathcal{N} = -2\pi N \int_{B_i^2}^{B_f^2} \frac{V}{V_{,I} B^2} dB^2 \approx \frac{\pi N B_0^2}{2} \frac{\frac{B_0^2}{B_i^2}}{\ln \frac{B_0^2}{B_i^2}}$$

where we assume $B_i \ll B_0$. In fact, the initial value B_i is determined by quantum fluctuations $B_i \sim H$ near the maximum $V \approx \frac{\lambda}{2} B_0^4$ which gives $B_i \approx B_0^2 \sqrt{\frac{4\pi}{3} N \lambda}$ implying $\frac{B_0^2}{B_i^2} = \frac{3}{4\pi \lambda N B_0^2}$. Neglecting the logarithmic dependence, the number of e-folds

$$\mathcal{N} \approx \frac{3}{8\lambda} \cdot \left(\ln \frac{3}{4\pi \lambda N B_0^2} \right)^{-1}$$

scales as λ^{-1} (in contrast to the logarithmic dependence for the symmetry breaking potential). For small values of λ the model can produce an arbitrary large number of e-folds of stable inflation with an approximately flat power spectrum of GW.

E. Exponential potential

Another example is provided by an exponential potential $V = \lambda \exp(-\alpha B^2)$. The small field assumption ($B \ll \frac{1}{\sqrt{N}}$) is not generally valid all the way until the end of inflation $B_f = \frac{\sqrt{6\pi N}}{\alpha}$, and one has to work in the limit of large fields. The inflation in such models is typically very long, and a large number of e-folds take place at negative values of β . For example, if we take $\alpha = \frac{N}{50}$ then the exit from inflation occurs at $B_f \approx \frac{217}{\sqrt{N}}$ and much more than the last 60 e-folds evolve at negative values of β . Such models would generically predict the absence of tensor modes in the CMB.

V. CONCLUSIONS

The recently proposed model of vector inflation [1] was shown to be very similar to the scalar field inflation at the background level. Despite of the apparent similarities a number of striking differences appear already at the first order in perturbation theory. The most important difficulty is the coupling of different modes (scalar, vector and tensor) even in the linear order. One can overcome the problem for the tensor perturbations in the limit of a large number of random fields with random fluctuations, when the coupling terms are statistically suppressed.

In this article we concentrated only on the behavior of gravitational waves on the homogeneous and isotropic background. Our analysis has shown that the behavior of tensor perturbations in vector inflation depends crucially on the form of the inflationary potential. The large field inflationary models generically lead to the tachyonic instabilities of GW (e.g. chaotic potential) and must be carefully analyzed. The stability of the small fields inflationary models is guaranteed whenever the slow-roll conditions are satisfied. In addition, the models of vector inflation can lead to an arbitrary tilt in the power spectrum as well as to a complete suppression of the tensor modes in the CMB.

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